ID Number: Key

Linear Test 1

1. (10 points) For each question, state if it is True or False. If True, briefly justify why it is true. If False, give an explanation or a counterexample to show why it is false.

   (a) Any augmented matrix with a bottom row of all zeroes represents a system that has an infinite number of solutions.

   False. Counter example:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

inconsistent $\Rightarrow$ no solutions

   (b) If a consistent system of linear equations has no free variables, then it has a unique solution.

   True. Consistent means there is either an ∞ # of solutions or a unique solution. No free variables $\Rightarrow$ not an ∞ # of solutions.

   (c) If v_1 is in span$\{v_2, v_3\}$, then v_2 is in span$\{v_3, v_1\}$.

   False. Let $v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   Then $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0v_2 + 0v_3 \in$ Span $\{v_2, v_3\}$,

   But $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not is span$\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ because it is not a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (d) If B is an $m \times n$ matrix with m pivots, then the linear transformation $T(\mathbf{x}) = \cancel{A}B\mathbf{x}$ is a one-to-one mapping.

   False:

$$\begin{array}{cc} & 3 \\ 2 & \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix} \end{array}$$

   Could have free variables and not be 1-1.

2. (15 points) For the system of equations

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 6 \\ -x_1 - 3x_3 &= 3 \\ 2x_2 - 4x_3 &= 0 \end{aligned}$$

do the following:

(a) Write the augmented matrix.

(b) Row reduce the matrix to reduced row echelon form.

(c) Find all solutions and write your answer in parametric vector form (if solutions exist).

(5) (a) $\left[\begin{array}{ccc|c} 2 & 4 & -2 & 6 \\ -1 & 0 & -3 & 3 \\ 0 & 2 & -4 & 0 \end{array}\right]$

(5) (b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ -1 & 0 & -3 & 3 \\ 0 & 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 2 & -4 & 6 \\ 0 & 2 & -4 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(5) (c) No solutions because the system is inconsistent.

3. (10 points) Let B be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose that $B\mathbf{y} = \mathbf{z}$. Is $B\mathbf{x} = 4\mathbf{z}$ consistent? Explain your answer.

Yes. Let $x = 4y$.

Then $Bx = B(4y)$ $\longleftarrow$ because B is linear

$= 4By$ $\longleftarrow$ Since $By = z$

$= 4z$

So $Bx = 4z$ has a solution (Namely, $x = 4y$)

thus the equation is consistent.

Also $\left[\begin{array}{c|c} B & z \end{array}\right]$ reduces to $\left[\begin{array}{ccc|c} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or some such

$\Rightarrow$ $\left[\begin{array}{c|c} B & 4z \end{array}\right]$ reduces to $\left[\begin{array}{ccc|c} 1 & & & 4* \\ & 1 & & 4* \\ & & 1 & 4* \\ & & & \end{array}\right]$

$\Rightarrow$ consistent.

4. ~~3~~. (15 points) Determine all values of p and q such that the system of equations

$$\begin{aligned} 2x_1 + px_2 &= 2 \\ 3x_1 + x_2 &= q \end{aligned}$$

has

(a) No solution.

(b) Infinitely many solutions.

(c) A unique solution.

$$\left[\begin{array}{cc|c} 2 & p & 2 \\ 3 & 1 & q \end{array}\right] \to \left[\begin{array}{cc|c} 1 & p/2 & 1 \\ 0 & 1-\frac{3p}{2} & q-3 \end{array}\right]$$ ✓✓✓

(a) There will be no solution when $1-\frac{3p}{2} = 0$ and $q-3 \neq 0$

4 $\Rightarrow 1 = \frac{3p}{2} \Rightarrow \boxed{p = \frac{2}{3} \text{ and } q \neq 3}$

(b) Infinitely many when $1-\frac{3p}{2} = 0$ and $q-3=0$

4 $\Rightarrow \boxed{p = \frac{2}{3} \text{ and } q = 3}$

(c) Unique solution when $1-\frac{3p}{2} \neq 0$

4 $\Rightarrow \boxed{p \neq \frac{2}{3}}$ q can be anything.

5. ~~4~~. (10 points) Are the vectors $v_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$ linearly dependent or linearly independent? Explain your answer.

Make them columns of matrix $\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ 2 & 8 & 5 \end{bmatrix}$ and row reduce

$\rightarrow A = \begin{bmatrix} 2 & 8 & 5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ is in echelon form

with a pivot in every column

$\Rightarrow$ no free variable so they are linearly independent.

(7)

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^2$? What about $\mathbb{R}^3$ or $\mathbb{R}^4$? Explain.

$\hookrightarrow$ No, because v_i are in $\mathbb{R}^3$, not $\mathbb{R}^2$.

(3)

Yes they span $\mathbb{R}^3$ because there is a pivot in every row of matrix A.

No they do not span $\mathbb{R}^4$ because they are in $\mathbb{R}^3$.

6. ~~5.~~ (15 points) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -11 \\ 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$.

(a) Find $T(u)$.

(b) Find all x in $\mathbb{R}^2$ whose image under T is b.

(c) Is c in the range of T? Explain your answer.

(5) (a) $$T(u) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 9+2 \\ 0-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -2 \end{bmatrix}$$

(5) (b) $$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -1 & -11 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -7 & -14 \\ 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Only one such x, $x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

(5) (c) $$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -7 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{array}\right] \leftarrow \text{inconsistent}$$

So c is <u>not</u> in the range of T

because the augmented matrix

is inconsistent.

7. (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the following transformation: T first lengthens vectors by a factor of 3, then projects vectors to the x_2 axis (a projection to the x_2 axis does the following- it sends x_1 to zero, and x_2 to x_2). Answer the questions below about T:

(a) For an arbitrary $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, what is $T(\mathbf{x})$?

(b) Show that T is a linear transformation.

(c) Find the standard matrix of T.

(d) Is T an onto map? Is T one-to-one? Explain.

(2) (a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{\text{lengthen by } 3} \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} \xrightarrow{\text{projection}} \begin{bmatrix} 0 \\ 3x_2 \end{bmatrix}$, so $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3x_2 \end{bmatrix}$

(6) (b) $T(u+v) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3(u_2+v_2) \end{bmatrix}$

$T(u)+T(v) = \begin{bmatrix} 0 \\ 3u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3u_2+3v_2 \end{bmatrix} =$

$T(cu) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3cu_2 \end{bmatrix}$

$c\,T(u) = c \cdot \begin{bmatrix} 0 \\ 3u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3cu_2 \end{bmatrix} =$

(3) (c) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

(4) (d) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \xrightarrow{\text{Row reduce}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a free variable and no pivot in bottom row.

$\Rightarrow T$ is <u>not</u> onto because there is not a pivot in every row

T is <u>not</u> 1-1 b/c there is a free variable.

8. ~~7~~. (10 points) If $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ 3 & 0 \end{bmatrix}$, find the matrix AB.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot 1 + 3\cdot 2 & -2\cdot 1 + 2\cdot 0 \\ 4\cdot 3 + 3\cdot(-2) & -2\cdot 3 + (-2)\cdot 0 \\ 4\cdot 0 + 3\cdot(-1) & -2\cdot 0 + 0\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -2 \\ 6 & -6 \\ -3 & 0 \end{bmatrix}$$

Ex. Cred:

A is 2×4

B is 4×3

C is 3×2